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Refined theories may be needed for vibration analysis of structures with overhang

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Abstract

The effect of shear deformation and rotary inertia terms on the free vibration of a beam with overhang was investigated. A recently proposed modified Timoshenko-type equations of motion were used to analyze the vibration of the structure. Two different sets of boundary conditions, with either a fixed or hinged end support, were studied. The results were compared with those obtained for the classical Bernoulli–Euler beam theory. The comparison shows that for a hinged end beam with very long overhang, where the span between the supports is less than one tenth of the overall beam length, the classical theory significantly overestimates the values of the fundamental natural frequencies, even for isotropic shear rigidity. On the other hand, the span effect is reversed for the clamped end beam, for which a relatively significant difference between the classical theory and shear theory results may occur only for a long span. For transversely isotropic beams, the refined theory predictions of the fundamental natural frequencies may be much smaller than those obtained through the rigid shear theory, especially for short span hinged end beams and long span clamped end beams. © 1999 Elsevier Science Ltd. All rights reserved.

1. Introduction

Bresse (1859) was apparently the first to publish a paper taking into account the rotary inertia in beam vibration, followed by Rayleigh (1878). Later, Timoshenko (1921) incorporated shear deformation effects. Since then many papers have been published on this subject. Grigoliuk and Selezov (1973) presented a most comprehensive worldwide review of bibliography, listing about 300 references dealing with refined theories of beams, plates and shells up to the year 1971. Recent developments were summarized in the monograph by Laura et al. (1992). In most cases, for isotropic structures at their lower end of natural frequencies, the effects of shear deformation and

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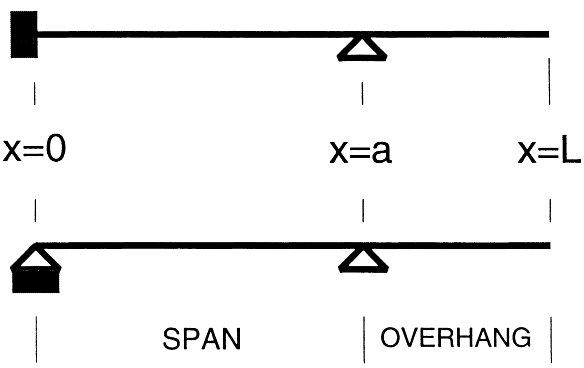

Fig. 1. Beam geometry and two sets of boundary conditions.

rotary inertia for solid cross-sections are insignificant. The effects turn out to be important for high frequency random excitation (Elishakoff and Lubliner, 1985; Elishakoff and Livshits, 1989), for short beams and for transversely isotropic materials (Brunelle, 1970). Recently, Panovko (1985) and Ari-Gur and Elishakoff (1990) have shown that the effects become of paramount importance for structures with overhang. In particular, the latter study, which considered the buckling of a column with overhang, showed that the significance of the shear effects depends on the span between the end support and the intermediate roller support of the beam. The present paper is an extension of that study to the case of free vibration.

2. Analysis

Consider a uniform beam of length L, which is either hinged or clamped at the end $x = 0$ and has a roller support at an intermediate location $x = a$ where $L \geqslant a > 0$, as presented in Fig. 1. The governing equations of dynamic equilibrium, following recent modifications proposed by Elishakoff (1994), are:

$$EI\psi_{,xx} + kAG(w_{,x} - \psi) = \rho I \ddot{w}_{,x} \tag{1}$$

$$kAG(w_{,x} - \psi)_{,x} = \rho A \ddot{w} \tag{2}$$

where EI and kGA are the bending and shear stiffnesses of the beam, respectively, incorporating the shear correction factor k of the cross-section, w is the lateral deflection, ψ is the rotation of the normal line, ρ is the mass density, $w_{,x} \equiv \partial w/\partial x$ and $\ddot{w} \equiv \partial^2 w/\partial t^2$.

Assuming solutions in the form:

$$\psi = \Psi(x)\,\mathrm{e}^{i\omega t}; \quad w = W(x)\,\mathrm{e}^{i\omega t} \tag{3}$$

the following coupled differential equations are obtained for the deflection shape $W(x)$ and the rotation angle $\Psi(x)$:

$$EI\Psi_{,xx} + (kAG + \rho I\omega^2)W_{,x} - kAG\Psi = 0 \tag{4}$$

$$kAGW_{,xx} - kAG\Psi_{,x} + \rho A\omega^2 W = 0 \tag{5}$$

In a non-dimensional form, defining the relative shear rigidity

$$\varepsilon \equiv \frac{kG}{E} \tag{6}$$

slenderness ratio of the beam

$$\lambda \equiv \frac{L}{r} \tag{7}$$

nondimensional fundamental frequency

$$\Omega \equiv \frac{\omega L}{c} \tag{8}$$

and relative deflection

$$\delta(\xi) \equiv \frac{W}{r} \tag{9}$$

where the radius of gyration r, wave propagation speed c and nondimensional axial coordinate ξ are, respectively

$$r \equiv \sqrt{\frac{I}{A}}; \quad c \equiv \sqrt{\frac{E}{\rho}}; \quad \xi \equiv \frac{x}{L} \tag{10}$$

the differential eqns (4)–(5) transform to:

$$\frac{d^2\Psi}{d\xi^2} + \left(\varepsilon\lambda + \frac{\Omega^2}{\lambda}\right)\frac{d\delta}{d\xi} - \varepsilon\lambda^2\Psi = 0 \tag{11}$$

$$\frac{d^2\delta}{d\xi^2} - \lambda\frac{d\Psi}{d\xi} + \frac{\Omega^2}{\varepsilon}\delta = 0 \tag{12}$$

2.1. Boundary conditions

As shown in Fig. 1, two different sets of boundary conditions are considered. The beam is either hinged or clamped at the end $\xi = 0$, the intermediate roller-type support is at $\xi = \alpha = a/L$ and the free end is at $\xi = 1$. The length of the overhang is then $L - a = (1-\alpha)L$.

The boundary conditions at the supported end $\xi = 0$ are:

$$\delta = 0 \quad \frac{d\Psi}{d\xi} = 0, \quad \text{hinged}$$

$$\delta = 0 \quad \Psi = 0, \quad \text{clamped} \tag{13}$$

for zero deflection and either zero bending moment (hinged end) or zero rotation (clamped end); at the free end $\xi = 1$ they are:

$$\frac{1}{\lambda}\frac{\mathrm{d}\delta}{\mathrm{d}\xi} - \Psi = 0; \quad \frac{\mathrm{d}\Psi}{\mathrm{d}\xi} = 0 \tag{14}$$

for zero shear force and bending moment; and at the intermediate support $\xi = \alpha$ the deflection is restrained ($\delta = 0$).

The continuity conditions at $\xi = \alpha$ read:

$$\begin{aligned} &\delta(\xi = \alpha^-) = \delta(\xi = \alpha^+) = 0 \\ &\Psi(\xi = \alpha^-) = \Psi(\xi = \alpha^+) \\ &\frac{\mathrm{d}\Psi}{\mathrm{d}\xi}(\xi = \alpha^-) = \frac{\mathrm{d}\Psi}{\mathrm{d}\xi}(\xi = \alpha^+) \end{aligned} \tag{15}$$

for continuity of deflection, rotation angle and internal bending moment.

2.2. Solution

The solutions for the coupled differential equations (11)–(12) are:

$$\delta = D_1 \sin(q\xi) + D_2 \cos(q\xi) + D_3 \sinh(p\xi) + D_4 \cosh(p\xi) \tag{16}$$

$$\Psi = P_1 \sin(q\xi) + P_2 \cos(q\xi) + P_3 \sinh(p\xi) + P_4 \cosh(p\xi) \tag{17}$$

where the coefficients are related through:

$$\begin{aligned} &\frac{P_2}{D_1} = -\frac{P_1}{D_2} = \frac{q^2 - \Omega^2/\varepsilon}{q\lambda} \\ &\frac{P_4}{D_3} = \frac{P_3}{D_4} = \frac{p^2 + \Omega^2/\varepsilon}{p\lambda} \end{aligned} \tag{18}$$

and:

$$q^2 = \sqrt{\left[\frac{\Omega^2}{2}\left(1 + \frac{1}{\varepsilon}\right)\right]^2 + \Omega^2\lambda^2} + \frac{\Omega^2}{2}\left(1 + \frac{1}{\varepsilon}\right) \tag{19}$$

$$p^2 = \sqrt{\left[\frac{\Omega^2}{2}\left(1 + \frac{1}{\varepsilon}\right)\right]^2 + \Omega^2\lambda^2} - \frac{\Omega^2}{2}\left(1 + \frac{1}{\varepsilon}\right) \tag{20}$$

The same general solutions [eqns (16)–(17)] apply for the entire length of the beam but with two different sets of coefficients D_i (and P_i) $\{i = 1, \ldots, 4\}$ for the ranges $0 \leqslant \xi \leqslant \alpha$ and $\alpha \leqslant \xi \leqslant 1$. With the relationships of eqn (18) there are in total eight independent coefficients. The boundary and continuity conditions, as described in eqns (13)–(15), provide eight homogeneous equations for these coefficients.

3. Results and discussion

The system of homogeneous equations which is obtained by applying the boundary conditions is of the form $[C]\{P\} = \{0\}$, where $\{P\}$ is a vector of eight coefficients P_i, four for the solution along the span and four for the overhang range. For non-trivial solution $\det[C] = 0$, which is the characteristic equation for the natural frequencies.

For the hinged end beam the coefficients P_1 and P_3 (or D_2 and D_4) for the span range vanish and $[C]$ is a 6×6 matrix, where $[C]$ for the clamped end beam remains an 8×8 matrix. The matrix $[C]$ for the hinged end beam is:

$$\begin{bmatrix} QSA & PSA & 0 & 0 & 0 & 0 \\ 0 & 0 & -QCA & QSA & PCA & PSA \\ 0 & 0 & AQS & AQC & APS & APC \\ 0 & 0 & QC & -QS & PC & PS \\ CQA & CPA & -SQA & -CQA & -SPA & -CPA \\ -SQQ & SPP & -CQQ & SQQ & -CPP & -SPP \end{bmatrix} \tag{21}$$

The matrix $[C]$ for the clamped end beam is:

$$\begin{bmatrix} -K1 & 0 & K2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -QCA & QSA & PCA & PSA & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -QCA & QSA & PCA & PSA \\ 0 & 0 & 0 & 0 & AQS & AQC & APS & APC \\ 0 & 0 & 0 & 0 & QC & -QS & PC & PS \\ SQA & CQA & SPA & CPA & -SQA & -CQA & -SPA & -CPA \\ CQQ & -SQQ & CPP & SPP & -CQQ & SQQ & -CPP & -SPP \end{bmatrix} \tag{22}$$

where

$$K1 = \frac{q\lambda}{q^2 - \Omega^2/\varepsilon}; \quad K2 = \frac{p\lambda}{p^2 + \Omega^2/\varepsilon} \tag{23}$$

$$QSA = K1\sin(q\alpha); \quad PSA = K2\sinh(p\alpha)$$

$$QCA = K1\cos(q\alpha); \quad PCA = K2\cosh(p\alpha) \tag{24}$$

$$\begin{Bmatrix} AQS \\ AQC \end{Bmatrix} = \begin{Bmatrix} \sin(q) \\ \cos(q) \end{Bmatrix} \left(K1\frac{q}{\lambda} - 1 \right) \tag{25}$$

$$\begin{Bmatrix} APS \\ APC \end{Bmatrix} = \begin{Bmatrix} \sinh(p) \\ \cosh(p) \end{Bmatrix} \left(K2\frac{p}{\lambda} - 1 \right) \tag{26}$$

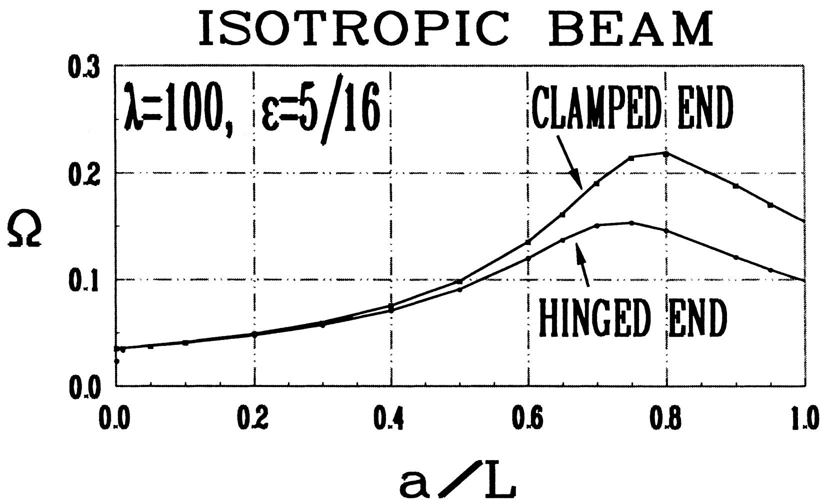

Fig. 2. Comparison between the present results (●) and the classical Bernoulli–Euler (—).

$$\begin{Bmatrix} QS \\ QC \\ PS \\ PC \end{Bmatrix} = \begin{Bmatrix} q\sin(q) \\ q\cos(q) \\ p\sinh(p) \\ p\cosh(p) \end{Bmatrix} \tag{27}$$

$$\begin{Bmatrix} SQA \\ CQA \\ SPA \\ CPA \end{Bmatrix} = \begin{Bmatrix} \sin(q\alpha) \\ \cos(q\alpha) \\ \sinh(p\alpha) \\ \cosh(p\alpha) \end{Bmatrix} \tag{28}$$

$$\begin{Bmatrix} SQQ \\ CQQ \\ SPP \\ CPP \end{Bmatrix} = \begin{Bmatrix} q\sin(q\alpha) \\ q\cos(q\alpha) \\ p\sinh(p\alpha) \\ p\cosh(p\alpha) \end{Bmatrix} \tag{29}$$

The characteristic equations and their roots were computed using symbolic computation software (MAPLE).

A comparison between the classical Bernoulli–Euler beam theory and the present refined theory results for an isotropic beam ($v = 1/3, G = 3E/8$) is presented in Fig. 2. It shows that the classical theory is inadequate for the hinged end beam with very short span between the supports.

For transversely isotropic beams, where the transverse shear rigidity is lower (as in honeycomb or foam core layered beams), the classical theory may be totally inadequate. Moreover, the type

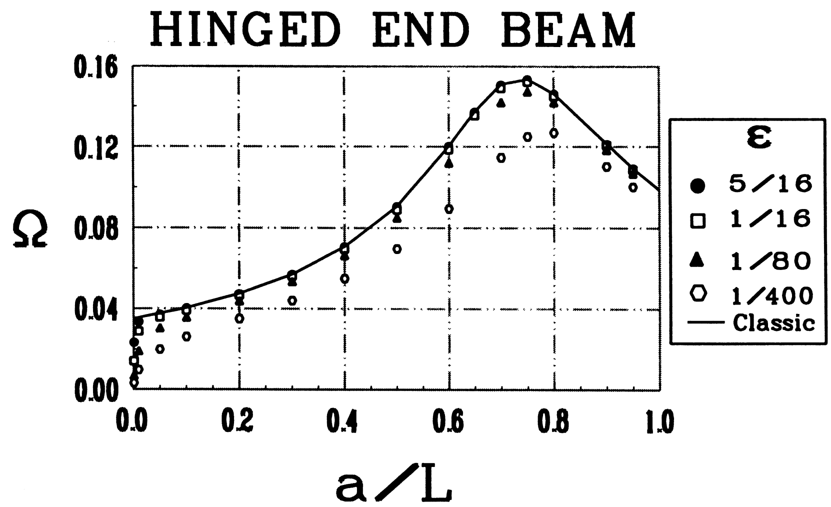

Fig. 3. Fundamental frequency of transversely isotropic hinged end beam vs span ratio for various shear rigidities.

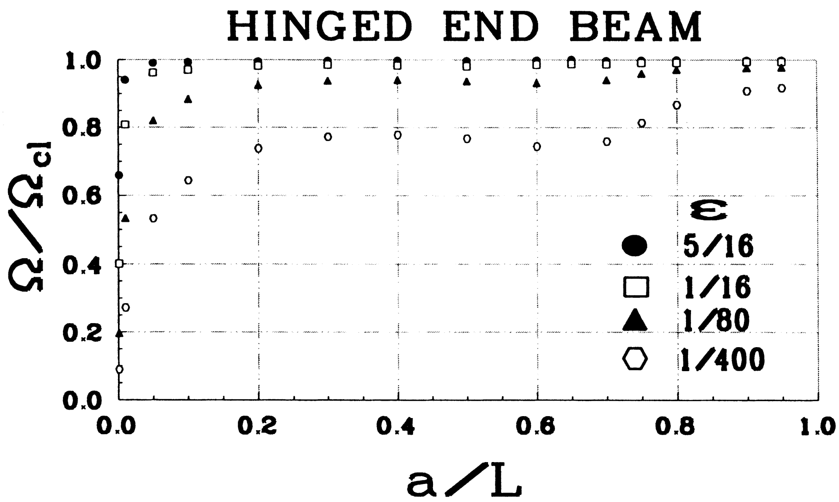

Fig. 4. Ratio of present theory to classical fundamental frequencies vs span ratio for various shear rigidities of hinged end beam.

of end conditions greatly affects the range of validity of classical beam results. Figure 3 presents natural frequencies for hinged end beams. Even for shear rigidities 25 times smaller than the isotropic ($\varepsilon \geqslant 1/80$) the differences in the fundamental frequencies are relatively small for spans of 10% or more ($a/L \geqslant 0.1$), but the differences become extremely large for shorter spans. To further demonstrate the inadequacy of the classical theory to predict the frequency of the short span hinged end beam, the ratio Ω/Ω_{cl} of the present theory fundamental frequency to the classical result is presented in Fig. 4. It shows a very low ratio for small a/L ratio. Note that large shear deformation

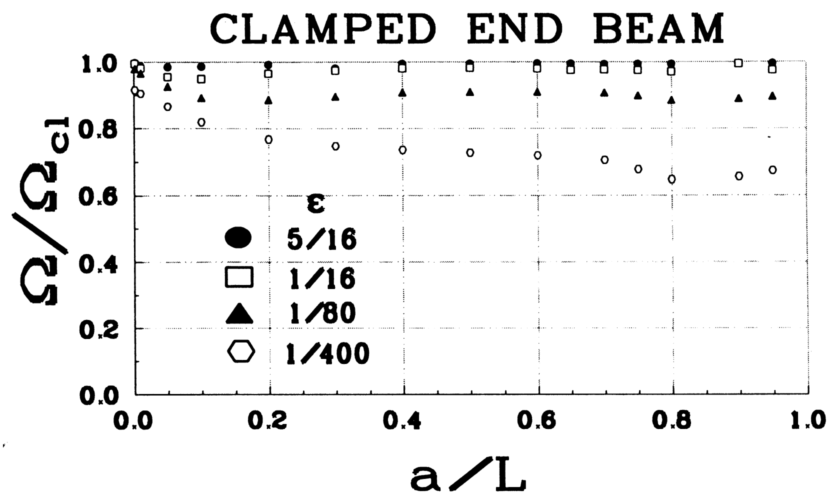

Fig. 5. Ratio of present theory to classical fundamental frequencies vs span ratio for various shear rigidities of clamped end beam.

effect for very short spans was also reported (Ari-Gur and Elishakoff, 1990) for column buckling of beams with similar boundary conditions.

Contrary to hinged end beams, the vibration of clamped end beams with very short spans ($\leqslant 10\%$) may be very well predicted by the classical beam theory. However, in this case the error due to neglecting shear deformation effects increases for large spans (see Fig. 5).

The differences in shear deformation between hinged end beams and clamped end beams are due to the differences in the relative shear reactions at the intermediate support. For hinged end beams, the intermediate support provides equilibrium and without it the beam is not balanced (a mechanism). To obtain a balancing moment, shorter spans require larger reaction forces and, as a result, larger shear forces and deformations are produced. On the other hand, clamped end beams are balanced even without the intermediate support. If this support is close to the clamped end it has a negligible effect, because it fixes an already fixed end. However, when moved toward the free end, it provides larger support and increases the stiffness of the beam. The increased support is in the form of a larger shear force and it produces larger shear deformations.

Another interesting phenomenon is the effect of shear rigidity on the optimum span, which is the span that results in the highest fundamental frequency. For hinged end beams the optimum is at $a/L = 0.75$ and it shifts to a higher span ratio when the shear rigidity is low (see Fig. 3). For clamped end beams the optimum span is $a/L = 0.8$ and it shifts to a shorter span for a low shear rigidity beam (see Fig. 6).

4. Conclusions

A solution for the free vibration problem of a beam with overhang was presented. A refined theory, including shear deformation and rotary inertia terms, was utilized. Numerical results for

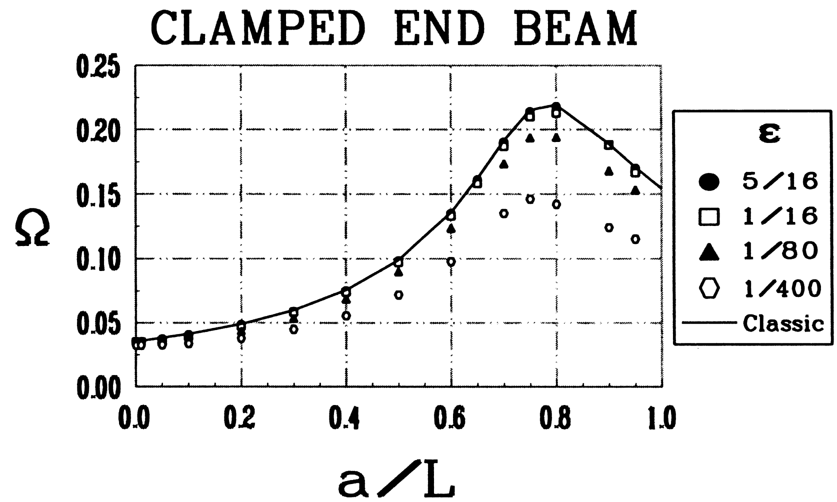

Fig. 6. Fundamental frequency of transversely isotropic clamped end beam vs span ratio for various shear rigidities.

two different sets of boundary conditions were presented and discussed. A comparison between the refined theory results and classical Bernoulli–Euler theory results demonstrated that for very short span between the supports the classical theory greatly overestimates the frequency of hinged end beams, but it is adequate for the prediction of the frequency of clamped end beams. This conclusion is essentially reversed for long span beams, where the error associated with the application of the classical theory is larger for clamped end beams. However, in this range of relatively short overhang, the refined theory is needed only for transversely isotopic beams with relatively low shear rigidity.

References

Ari-Gur, J., Elishakoff, I., 1990. On the effect of shear deformation on buckling of columns with overhang. Journal of Sound and Vibration 139 (1) 165–169.

Bresse, M., 1859. Cours de Mechanique Appliquee. Mallet-Bachelier, Paris (in French).

Brunelle, E.J., 1970. Elastic stability of transversely isotropic Timoshenko beams. AIAA Journal 8, 2271–2273.

Elishakoff, I., 1994. An equation which is simpler and more consistent than Bresse–Timoshenko equations. Report, Florida Atlantic University.

Elishakoff, I., Livshits, D., 1989. Some closed-form solutions in random vibration of Timoshenko beams. Probabilistic Engineering Mechanics 4 (1) 49–54.

Elishakoff, I., Lubliner, E., 1985. Random vibration of a structure via classical and non-classical theories. In: Eggwertz S., Lind, N. (Eds.), Probabilistic Methods in the Mechanics of Solids and Structures. Springer, Berlin, pp. 455–468.

Grigoliuk, E.I., Selezov, I.T., 1973. Nonclassical Theories of Vibration for Beams, Plates and Shells. Viniti, Moscow (in Russian).

Laura, P.A.A., Grossi, R.E., Maurizi, M.J., 1992. Vibrating Timoshenko Beams. Institute of Applied Mechanics, Bahia Blanca, Argentina.

Panovko, Y.G., 1985. Mechanics of Deformable Solids. Nauka, Moscow (in Russian).

Rayleigh, J.W.S., 1878. The Theory of Sound. Macmillan, London.

Timoshenko, S.P., 1921. On the correction for shear of the differential equation for transverse vibrations of prismatic bar. Philosophical Magazine 6 (41/245), 744–746.